Analytical description of the radiative-conductive heat transfer in a gray medium

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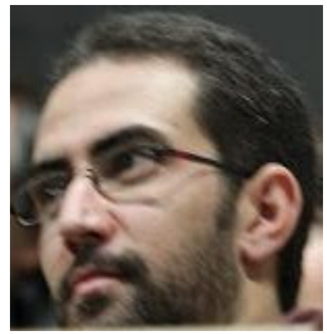

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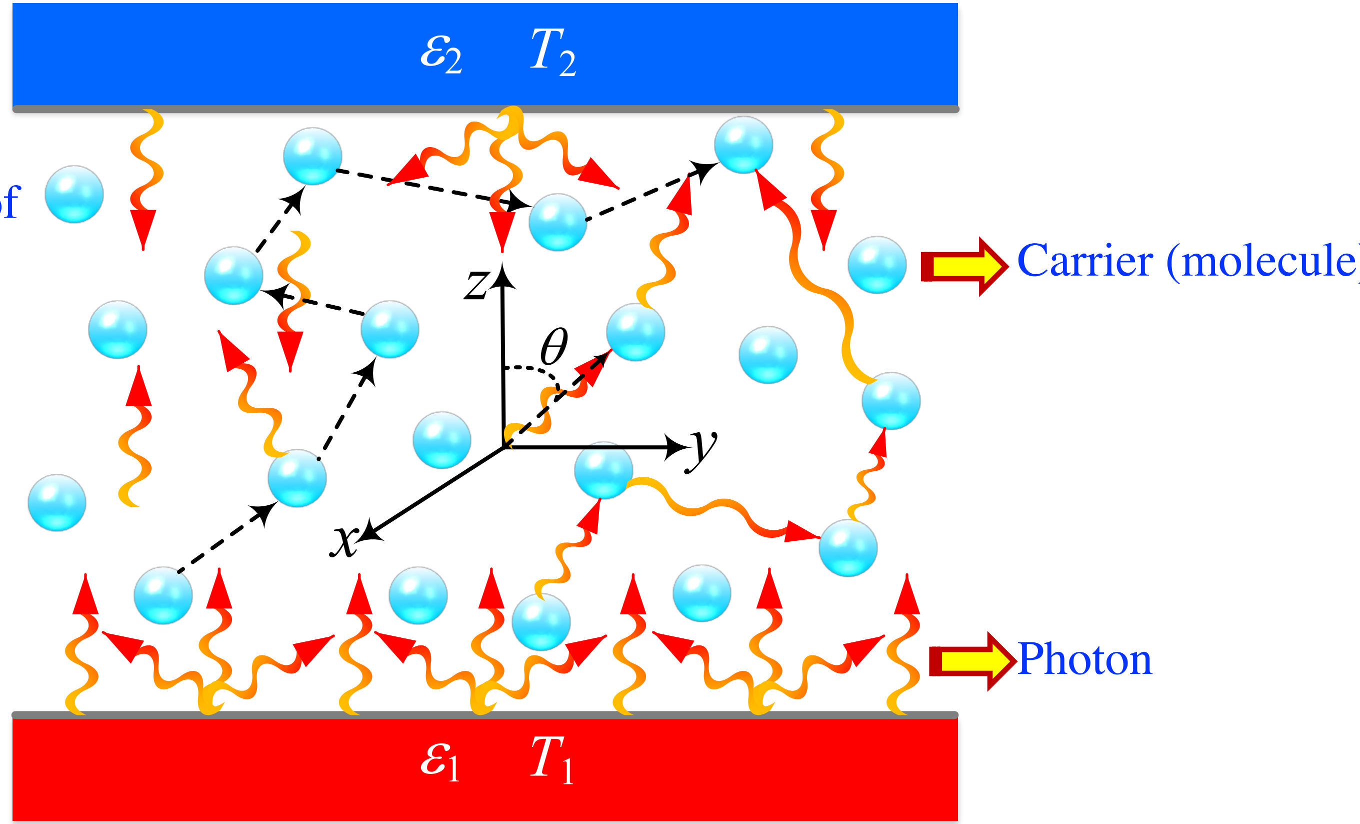
1. Problem formulation

Gray medium contained between two diffuse/gray parallel plates.

Absorption and scattering coefficients independent of the spectral frequency.



Heat transfer by **photons** as well as by **photon-carrier** and **carrier-carrier** interactions

Radiation | Radiation-Conduction | Conduction

Goal of our work:

To analytically solve for:

Heat flux = $q = q_r + q_c = ?$

Temperature = $T(z) = ?$

Radiative Transfer Equation (RTE)

$$I_0(\xi) = n^2 \sigma T^4(\xi)/\pi$$

Radiation intensity

Blackbody intensity

$$\xi = \kappa z$$

$$\kappa = \kappa_a + \kappa_s$$

$$I + \mu \frac{\partial I}{\partial \xi} = S(\xi) = (1-\Omega) I_0(\xi) + \frac{\Omega}{2} \int_{-1}^{1} I(\xi, \mu) d\mu \qquad \mu = \cos(\theta)$$

Extintion coefficient

Source

Scattering albedo: $\Omega = \kappa_s/\kappa$

Radiative heat flux: $$q_r(\xi) = 2\pi \int_{-1}^{1} I(\xi, \mu) \mu d\mu$$

Temperature:
(Obtained after integrating the RTE)

$$S(\xi) = \frac{1}{2} \int_{-1}^{1} I(\xi, \mu) d\mu + \frac{1}{4\pi} q_r'(\xi)$$
$$= I_0(\xi) - \frac{\Omega}{4\pi(1-\Omega)} q_r'(\xi),$$

Conductive heat flux: $$q_c(\xi) = -\kappa k dT/d\xi$$

In absence of heat conduction:

$$S(\xi) = I_0(\xi)$$

(Chandrasekhar's result)

Total heat flux: $$q_r(\xi) + q_c(\xi) = \text{constant}$$

(Principle of energy conservation)

Outline

1. Integral solution of the RTE

Formalism reported by Modest

2. Pure radiative heat transfer

Discrete ordinates method (DOM)

Integral solution+DOM

Results for $T(z)$ et q

3. Radiative-conductive heat transfer

DOM

Integral solution+DOM

Results for $T(z)$ et q

1. Integral solution of the RTE

$$I^+(\xi,\mu) = \frac{J_1^+}{\pi}e^{-\xi/\mu} + \frac{1}{\mu}\int_0^{\xi} S(\xi')e^{-(\xi-\xi')/\mu}d\xi', \Rightarrow \quad 0<\mu<1$$

$$I^-(\xi,\mu) = \frac{J_2^-}{\pi}e^{(\tau-\xi)/\mu} - \frac{1}{\mu}\int_{\xi}^{\tau} S(\xi')e^{(\xi'-\xi)/\mu}d\xi' \Rightarrow -1<\mu<0$$

Radiosities
$$\begin{cases} J_1^+ = \pi I(0,\mu) \\ J_2^- = \pi I(\tau,\mu) \end{cases}$$

Radiative heat fluxes leaving the diffuse gray surfaces

$\xi = 0 \qquad \xi = \tau = \kappa d$

Independent of μ.

Radiative heat flux:

$$\frac{q_r(\xi)}{2} = J_1^+E_3(\xi) - J_2^-E_3(\tau-\xi) + \int_0^{\xi} J(\xi')E_2(\xi-\xi')d\xi' - \int_{\xi}^{\tau} J(\xi')E_2(\xi'-\xi)d\xi',$$

Temperature:

$J(\xi) = \pi S(\xi)$

$$2J(\xi) = \frac{1}{2}q_r'(\xi) + J_1^+E_2(\xi) + J_2^-E_2(\tau-\xi) + \int_0^{\tau} J(\xi')E_1(|\xi'-\xi|)d\xi'$$

$$E_n(a) = \int_0^1 x^{n-2}e^{-a/x}dx$$

Normalized heat flux and temperature

Following Modest's book: $Q_r(\xi) = \dfrac{q_r(\xi)}{J_1^+ - J_2^-}$ $\quad U(\xi) = \dfrac{J(\xi) - J_2^-}{J_1^+ - J_2^-}$

Radiative heat flux: $$\frac{Q_r(\xi)}{2} = E_3(\xi) - \frac{d}{d\xi}\int_0^{\tau} U(\xi')E_3(|\xi' - \xi|)d\xi'$$

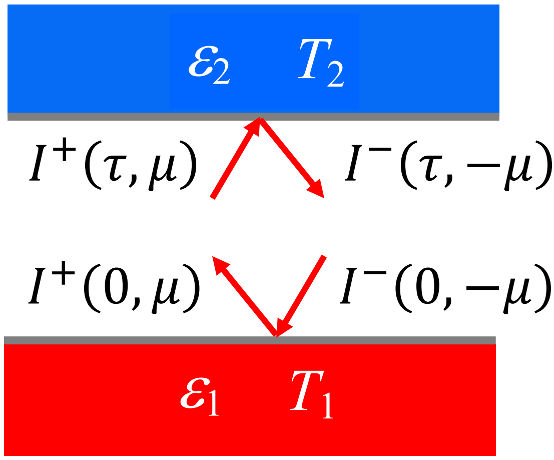
Temperature: $$2U(\xi) = \frac{1}{2}Q_r'(\xi) + E_2(\xi) + \int_0^{\tau} U(\xi')E_1(|\xi' - \xi|)d\xi'$$

Fredholm's integral equation (unknown analytical solution)

Approximate analytical methods	**Numerical methods**
• Optically thin/thick approx.	• Successive approximations
• Schuster-Schwarzschild approx.	• Spherical harmonics
• Milne-Eddington approx.	• Discrete ordinates
• Moment/variational methods	• Monte Carlo
• Exponential kernel approx.	

Boundary Conditions

Energy balance



$$2\pi\int_0^1 I^-(\tau,-\mu)\mu d\mu = (1-\varepsilon_2)2\pi\int_0^1 I^+(\tau,\mu)\mu d\mu + \varepsilon_2 J_0(\tau)$$

$$2\pi\int_0^1 I^+(0,\mu)\mu d\mu = (1-\varepsilon_1)2\pi\int_0^1 I^-(0,-\mu)\mu d\mu + \varepsilon_1 J_0(0)$$

$$J_0(\xi) = \pi I_0(\xi) = n^2\sigma T^4$$

$$\Rightarrow \begin{bmatrix} \varepsilon_1+\psi_1 & -\psi_1 \\ -\psi_2 & \varepsilon_2+\psi_2 \end{bmatrix}\begin{bmatrix} J_1^+ \\ J_2^- \end{bmatrix} = \begin{bmatrix} \varepsilon_1 J_0(0) \\ \varepsilon_2 J_0(\tau) \end{bmatrix} \qquad \begin{matrix} \psi_1 = (1-\varepsilon_1)Q_r(0) \\ \psi_2 = (1-\varepsilon_2)Q_r(\tau) \end{matrix}$$

Radiative heat flux: $\dfrac{q_r(\xi)}{J_0(0)-J_0(\tau)} = \dfrac{Q_r(\xi)}{f(\tau)}$ (Flux)

Temperature: $\dfrac{J_0(\xi)-J_0(\tau)}{J_0(0)-J_0(\tau)}f(\tau) = U(\xi) + \dfrac{\Omega}{4(1-\Omega)}Q_r'(\xi) + (\varepsilon_2^{-1}-1)Q_r(\tau)$ (Temp)

$$f(\tau) = 1 + (\varepsilon_1^{-1}-1)Q_r(0) + (\varepsilon_2^{-1}-1)Q_r(\tau)$$

Remaining problem: $U(\xi)=?$ $Q_r(\xi)=?$

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Modest's Formalism

2. Pure radiative heat transfer

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Integral solution+DOM

Results for $T(z)$ et q

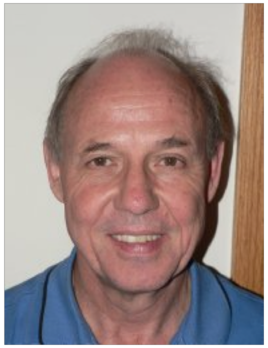
2. Pure radiative heat transfer (vacuum problem)

No heat transfer by the interactions of the medium energy carriers

$$\frac{q}{n^2\sigma\left(T_1^4 - T_2^4\right)} = \frac{Q}{1 + (\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2)Q}$$

$Q_r(\xi) = Q$ = constant

$$\frac{T^4(\xi) - T_2^4}{T_1^4 - T_2^4} = \frac{U(\xi) + (\varepsilon_2^{-1} - 1)Q}{1 + (\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2)Q}$$



$$\frac{Q_r(\xi)}{2} = E_3(\xi) - \frac{d}{d\xi}\int_0^{\tau} U(\xi')E_3(|\xi' - \xi|)d\xi'$$

$$2U(\xi) = E_2(\xi) + \int_0^{\tau} U(\xi')E_1(|\xi' - \xi|)d\xi'$$

Eqs. derived by M. Modest.

Limiting solutions:

Optically thin medium ($\tau \ll 1$): $Q^{-1} = 1 + 3\tau/4$ Modest, Majumdar, Chen.

Optically thick medium ($\tau \gg 1$): $Q = (4/3)/(1.4210 + \tau)$ Heaslet and Warming

Discrete ordinates method (DOM)

This method developed by Chandrasekhar is based on the Gaussian quadrature:

$$\int_{-1}^{1} F(\mu)d\mu = \sum_n a_n F(\mu_n) \Rightarrow F(\mu) = \mu^i \Rightarrow \sum_n a_n \mu_n^i = \begin{cases} \frac{2}{1+i}, & i = 0, 2, 4, \ldots \\ 0, & i = 1, 3, 5, \ldots \end{cases}$$

$$a_n = a_{-n}$$
$$\mu_{-n} = -\mu_n$$

Split of the integration interval −1≤μ≤1 of an arbitrary function $F(\mu)$ in $2N$ symmetrical directions.

RTE ⇒

$$I(\xi, \mu_m) + \mu_m \frac{\partial I(\xi, \mu_m)}{\partial \xi} = \frac{1}{2} \sum_n a_n I(\xi, \mu_n) \qquad m = \pm 1, \pm 2, \ldots, \pm N$$

System of $2N$ linear differential equations

⇓ Solution first derived by Chandrasekhar.

$$I(\xi, \mu) = \alpha' \left[\xi + \beta' - \mu + \sum_{j=1}^{N-1} D_j \left(\frac{e^{-\delta_j \xi}}{1 - \mu\delta_j} - \frac{e^{-\delta_j(\tau - \xi)}}{1 + \mu\delta_j} \right) \right]$$

DOM solution for the temperature and heat flux

Temperature equation

$$2I_0(\xi) = \int_{-1}^{1} I(\xi,\mu)d\mu$$

$$2I_0(\xi) = \sum_n a_n I(\xi,\mu_n)$$

$$I_0(\xi) = \alpha'\left[\xi + \beta' + \sum_{j=1}^{N-1} D_j\left(e^{-\delta_j\xi} - e^{-\delta_j(\tau-\xi)}\right)\right]$$

Heat flux

$$q(\xi) = 2\pi\int_{-1}^{1} I(\xi,\mu)\mu d\mu$$

$$q(\xi) = 2\pi\sum_n a_n I(\xi,\mu_n)\mu_n$$

$$q(\xi) = -\frac{4\pi}{3}\alpha'$$

Series expansion of an unknown function $p(\xi)$.

$$I_0(\xi) = \alpha'[\xi + \beta' + \gamma(p(\xi) - p(\tau-\xi))]$$

$$U(\xi) = (\pi I_0(\xi) - J_2^-)/(J_1^+ - J_2^-)$$

$$\begin{cases} U(\xi) = 1 - \alpha[\xi + \beta + \gamma(p(\xi) - p(\tau-\xi))] \\ \alpha = \frac{3}{4}Q = \frac{1}{\tau+2\beta} \end{cases}$$

Independent of position!

General solutions of the Fredholm integral equations in terms of the parameters $\beta(\tau)$ and $\gamma = \gamma(\tau)$ as well as of the function $p(\xi)$.

Solutions for $\beta(\tau)$, $\gamma = \gamma(\tau)$, and $p(\xi)$

Fredholm integral equations at $\xi = 0$:

$$Q = 1 - 2\int_0^\tau U(\xi')E_2(\xi')d\xi' \qquad 2U(0) = 1 + \int_0^\tau U(\xi')E_1(\xi')d\xi'$$

$$\begin{bmatrix} 1 - E_2(\tau) & \chi \\ 1/2 + E_3(\tau) & C_2 \end{bmatrix} \begin{bmatrix} \beta(\tau) \\ \gamma(\tau) \end{bmatrix} = \begin{bmatrix} 1/2 - E_3(\tau) \\ 1/3 + E_4(\tau) \end{bmatrix} \qquad \textbf{(1)}$$

$$\chi = 2(p(0) - p(\tau)) - C_1 \qquad C_n = \int_0^\tau [p(\xi) - p(\tau - \xi)]E_n(\xi)d\xi$$

Fredholm integral equation for the temperature at $\xi \to \infty$:

$$\gamma_\infty p(\xi) = -\beta_\infty E_2(\xi) + E_3(\xi) + \gamma_\infty \int_0^\tau p(\xi')E_1(|\xi' - \xi|)d\xi' \qquad \textbf{(2)}$$

$$\gamma_\infty p(\xi) = b_0 E_2(\xi) + c_0 E_3(\xi)$$

Key assumption!

Decomposition on the base of E_2, E_3, …

From (1) and (2): $\gamma_\infty = 1$

$$\beta_\infty + (2 - I_{12})b_0 + (1 - I_{13})c_0 = \frac{1}{2} \qquad \frac{\beta_\infty}{2} + I_{22}b_0 + I_{23}c_0 = \frac{1}{3} \qquad \frac{\beta_\infty}{3} - I_{23}b_0 - I_{33}c_0 = \frac{1}{4}$$

$$I_{nm} = \int_0^\infty E_n(\xi)E_m(\xi)d\xi \qquad \beta_\infty = 0.71047,\ b_0 = -0.25082 \text{ and } c_0 = 0.23526$$

Summary of the solution

$$\frac{q}{n^2\sigma\left(T_1^4 - T_2^4\right)} = \frac{Q}{1 + (\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2)Q}$$

$$\frac{T^4(\xi) - T_2^4}{T_1^4 - T_2^4} = \frac{U(\xi) + (\varepsilon_2^{-1} - 1)Q}{1 + (\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2)Q}$$

$$\begin{cases} U(\xi) = 1 - \alpha[\xi + \beta + \gamma(p(\xi) - p(\tau - \xi))] \\ \alpha = \frac{3}{4}Q = \frac{1}{\tau + 2\beta} \end{cases}$$

$$\begin{bmatrix} 1 - E_2(\tau) & \chi \\ 1/2 + E_3(\tau) & C_2 \end{bmatrix} \begin{bmatrix} \beta(\tau) \\ \gamma(\tau) \end{bmatrix} = \begin{bmatrix} 1/2 - E_3(\tau) \\ 1/3 + E_4(\tau) \end{bmatrix}$$

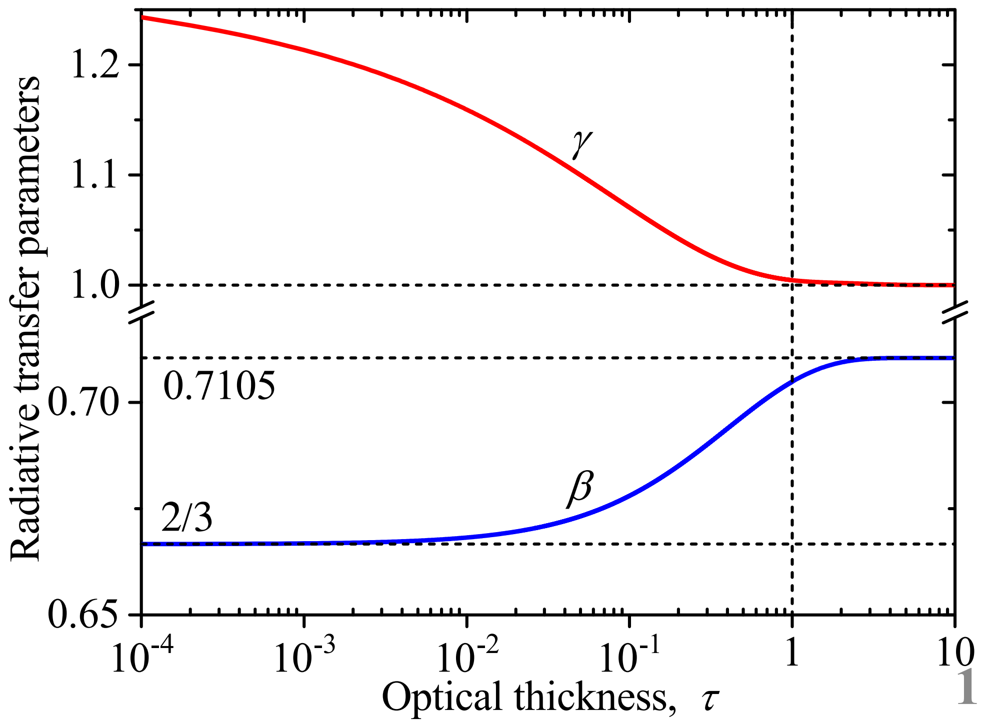
$$p(\xi) = b_0 E_2(\xi) + c_0 E_3(\xi)$$

$b_0 = -0.25082$ and $c_0 = 0.23526$

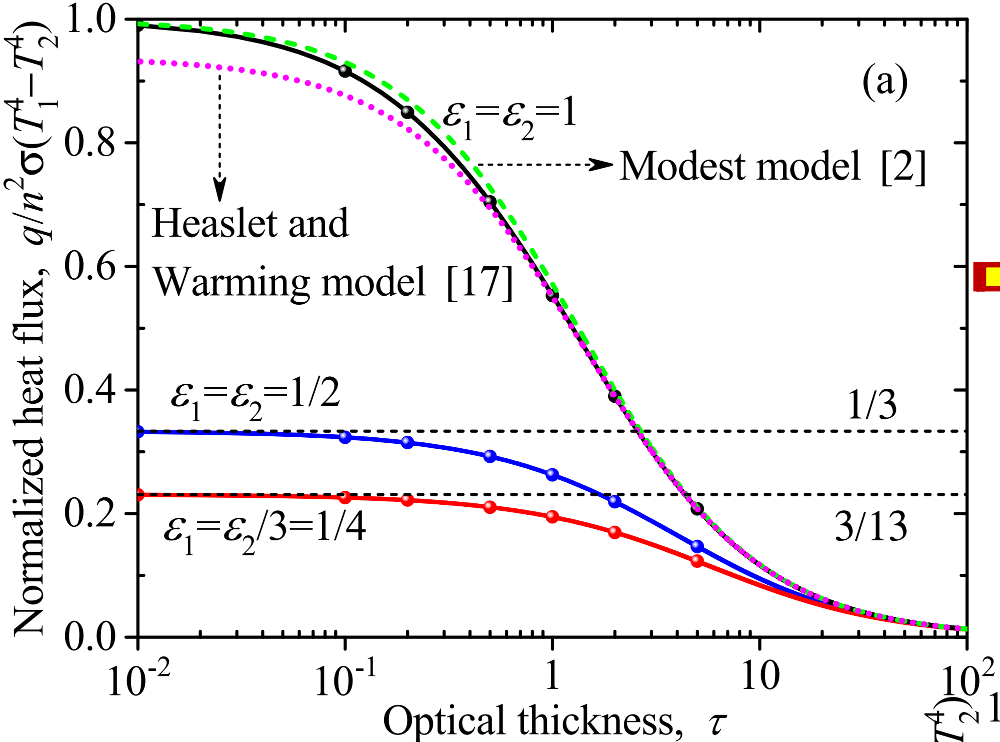
Remark: $q = k_{eff}(T_1 - T_2)/d$

$$k_{eff} = \frac{n^2\sigma(T_1 + T_2)(T_1^2 + T_2^2)d}{\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2 + 3(2\beta + \tau)/4}$$

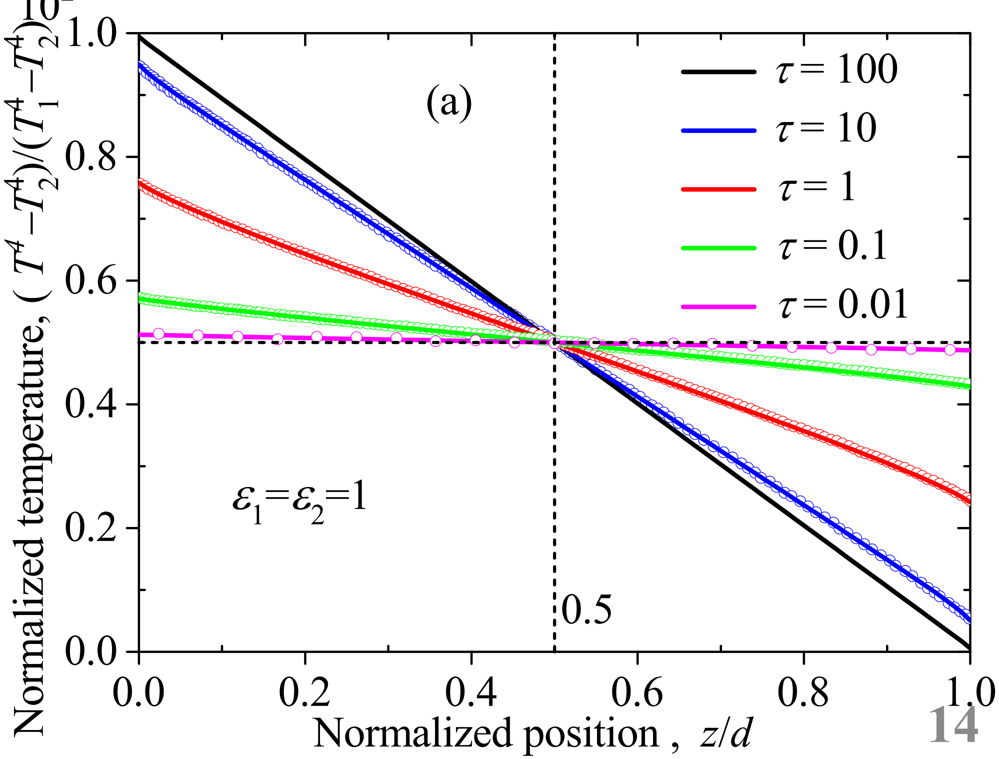
Effective radiative conductivity



Solutions 1

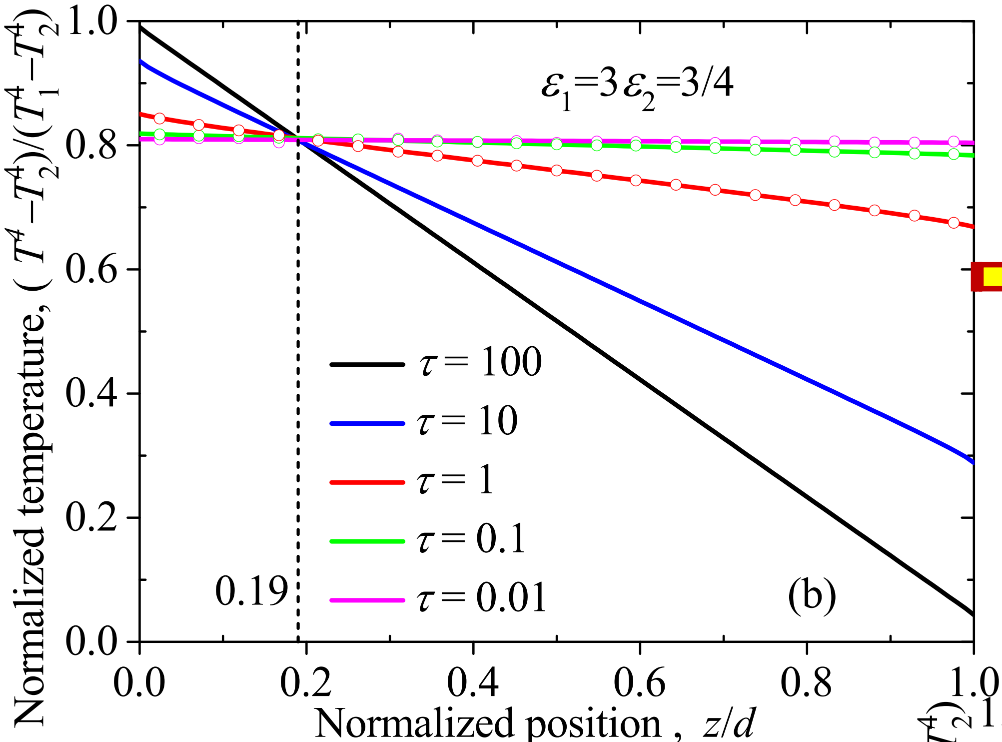


Excellent agreement with the numerical predictions (dots) of the Monte Carlo method, for different emissivities.

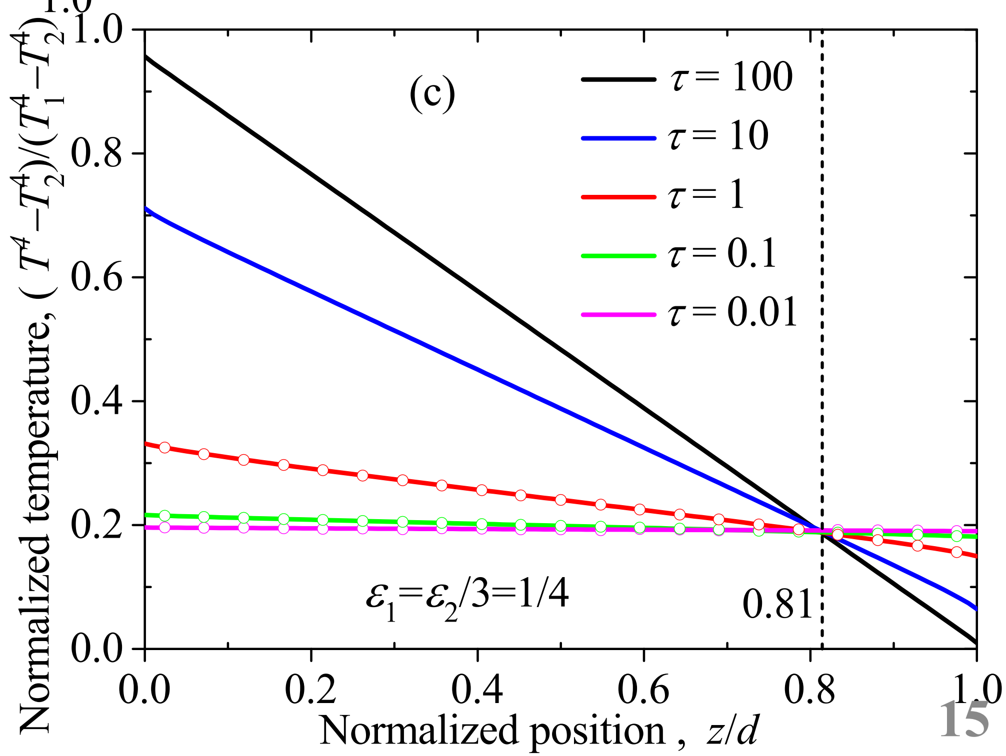


Excellent agreement with the predictions (dots) of the Monte Carlo method, for a large interval of optical thicknesses $\boldsymbol{\tau}$.

Solution 2



Excellent agreement with the numerical predictions (dots) of the Monte Carlo method.



Excellent agreement with the numerical predictions (dots) of the Monte Carlo method.

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3. Radiative-conductive heat transfer

Discrete ordinates solution:

$$U(\xi) = 1 - A[\xi + B + C\Omega(p_{23}(\xi) - p_{23}(\tau - \xi))] \qquad A^{-1} = \tau + 2B$$

$$Q_r(\xi) = \frac{4A}{3}[1 - 3(1-\Omega)C(p_{34}(\xi) + p_{34}(\tau - \xi))] \qquad p_{23}(\xi) = -p'_{34}(\xi)$$

From integral equations:

$$\begin{bmatrix} 1 - E_2(\tau) & \chi \\ 1/2 + E_3(\tau) & \Gamma \end{bmatrix} \begin{bmatrix} B(\tau) \\ C(\tau) \end{bmatrix} = \begin{bmatrix} 1/2 - E_3(\tau) \\ 1/3 + E_4(\tau) \end{bmatrix}$$

$$\boxed{p_{23}(\xi) = bE_2(\xi) + cE_3(\xi)} \qquad \boxed{p_{34}(\xi) = bE_3(\xi) + cE_4(\xi)}$$

$$\begin{bmatrix} 1 & 2 - \Omega I_{12} & 1 - \Omega I_{13} \\ 1/2 & 1 - \Omega + \Omega I_{22} & \frac{2}{3}(1-\Omega) + \Omega I_{23} \\ 1/3 & \frac{2}{3}(1-\Omega) - \Omega I_{23} & \frac{1-\Omega}{2} - \Omega I_{33} \end{bmatrix} \begin{bmatrix} B_\infty \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$$

$$I_{nm} = \int_0^\infty E_n(\xi) E_m(\xi) d\xi$$

$$\begin{cases} b_0 = b(\Omega = 1) \\ c_0 = c(\Omega = 1) \end{cases}$$

Heat flux and temperature profiles 1

From Eqs. (Flux) and (Temp):

$$\frac{q_r(\xi)}{n^2\sigma\left(T_1^4 - T_2^4\right)} = \frac{Q_r(\xi)}{f(\tau)}$$

$$\frac{T^4(\xi) - T_2^4}{T_1^4 - T_2^4} f(\tau) = 1 - A(\xi + B) + (\varepsilon_2^{-1} - 1)Q_r(\tau)$$

$$f(\tau) = 1 + (\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2)Q_r(\tau)$$

Corrected temperature profile:

$$q_c(\xi) = -k\kappa\, dT(\xi)/d\xi = q_t - q_r(\xi) = q_t - (J_0(0) - J_0(\tau))Q_r(\xi)/f(\tau)$$

Fourier's law

Constant (Energy conservation)

$$T(\xi) = T_0 - \frac{q_t \xi}{k\kappa} + \frac{4An^2\sigma(T_1^4 - T_2^4)}{3k\kappa f(\tau)}[\xi + 3\Psi(\xi)]$$

Integration constant

$$\Psi(\xi) = (1-\Omega)C(p_{45}(\xi) - p_{45}(\tau - \xi)) \qquad p_{45}(\xi) = bE_4(\xi) + cE_5(\xi).$$

Boundary conditions:

$$T(0) = T_1 \qquad T(\tau) = T_2$$

Heat flux and temperature profiles 2

Total heat flux

$$q_t = \underbrace{\frac{k}{d}(T_1 - T_2)}_{\text{Conduction}} + \underbrace{\frac{4}{3f(\tau)}\frac{n^2\sigma(T_1^4 - T_2^4)}{\tau + 2B}}_{\text{Radiation}}\left(1 + \underbrace{6\frac{\Psi(\tau)}{\tau}}_{\text{Coupling}}\right)$$

Temperature

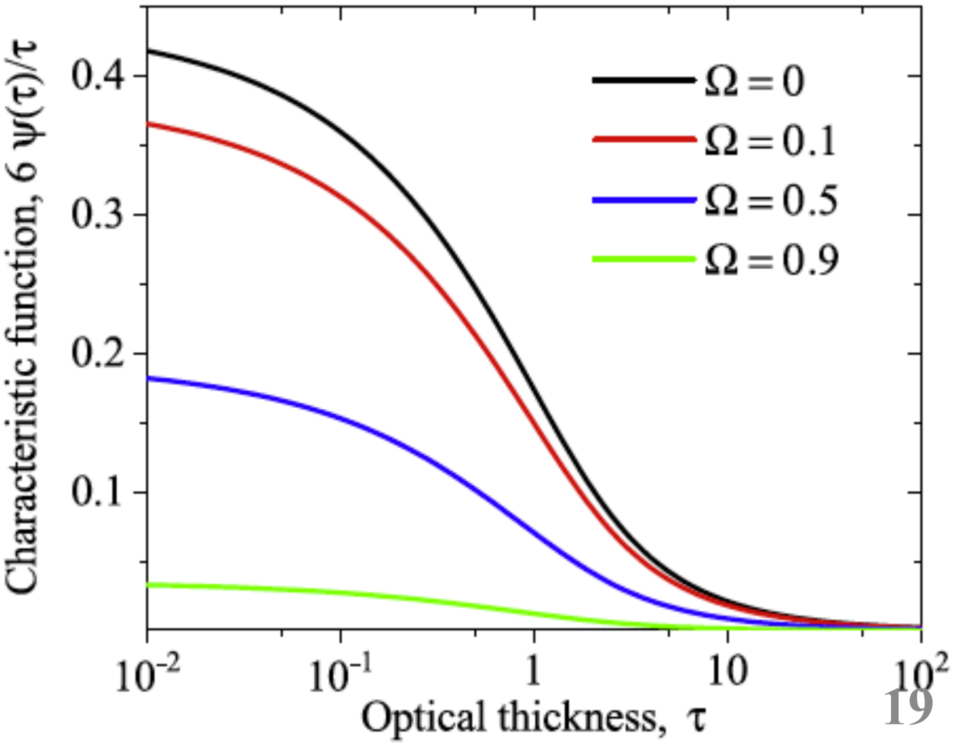
$$T(\xi) = T_1 - (T_1 - T_2)\frac{\xi}{\tau} + \frac{4}{k\kappa f(\tau)}\frac{n^2\sigma(T_1^4 - T_2^4)}{\tau + 2B}\times\left[\Psi(\xi) + \left(1 - \frac{2\xi}{\tau}\right)\Psi(\tau)\right]$$

Effective thermal conductivity

$$K_{eff} = k + \frac{n^2\sigma(T_1 + T_2)(T_1^2 + T_2^2)d(1 + 6\Psi(\tau)/\tau)}{(\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2)(1 - 3\chi) + 3(2\beta + \tau)/4}$$

Relevant for optically thin media without scattering:

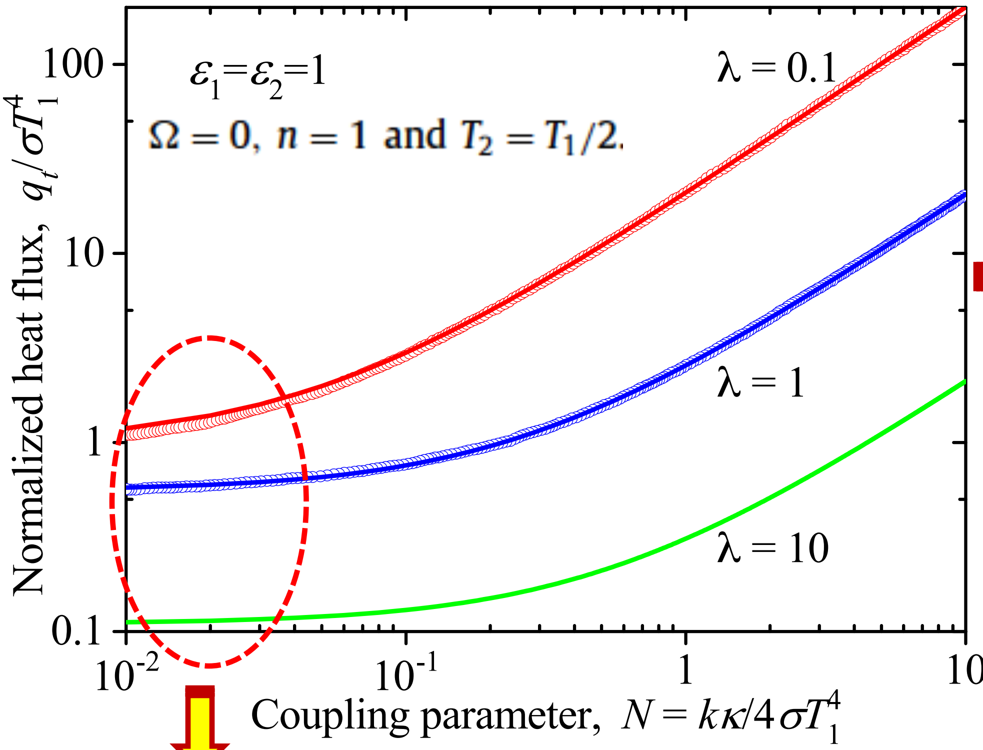
$\Omega = 0$



In absence of absorption (Ω=1):

$\Psi = 0$

Results

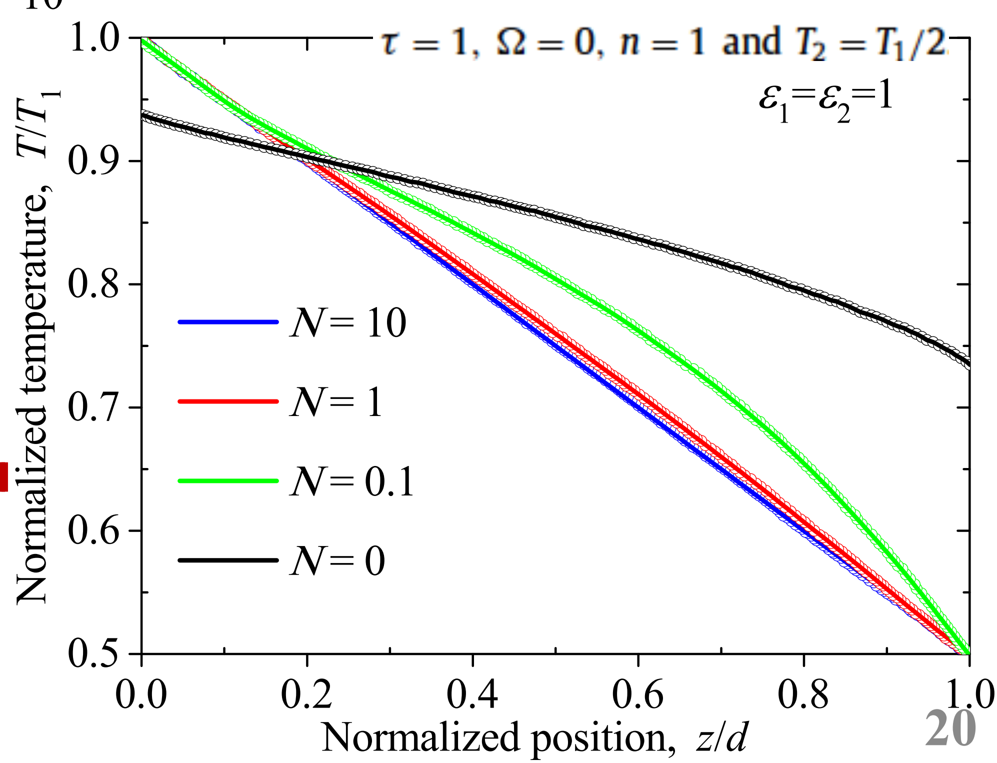


Excellent agreement with the numerical predictions (dots) reported by Modest.

Accuracy > 99.9%

$$(1 - |1 - S_{\mathrm{ana}}/S_{\mathrm{num}}|)x100\%$$

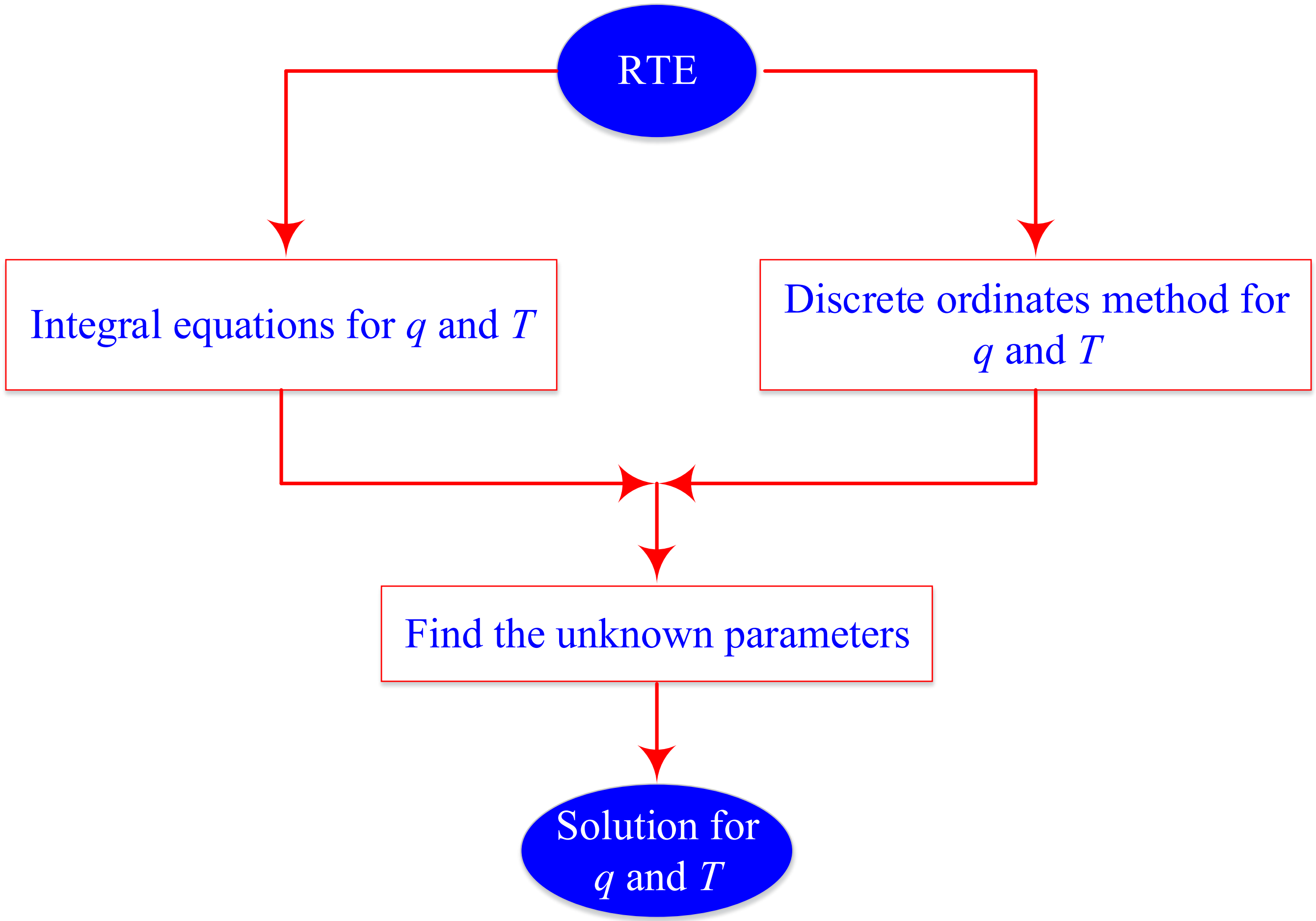
Weak heat conduction: Total heat flux nearly independent of *N*.



- Boundary discontinuities in absence of heat conduction (N = 0) only.
- Excellent agreement with the numerical predictions (dots) reported by Modest.

Conclusion to take home

The discrete ordinates method can be used to analytically solve the RTE with an accuracy higher than 99.9%.



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Thank you!

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